The Simplification of the Arithmetical Processes of Involution and Evolution.

By E. Chappell, City and Guilds (Engineering) College.

(Communicated by Prof. W. E. Dalby, F.R.S. Received January 25, 1915.)

It is a self-evident statement that addition and subtraction are the most simple arithmetical processes, so that any other process may be said to be completely simplified when it is replaced by either of these two. The invention of logarithms completely simplified multiplication and division, but only reduced involution and evolution to multiplication and division. In modern applied science there are many laws and empirical formulæ in which fractional indices occur, and the calculations sometimes become troublesome on this account. Although the obvious course is to perform the necessary multiplication or division of the logarithms, in such cases, by the addition or subtraction of *their* logarithms, no one seems to have considered it worth while to construct a Table giving the logarithms of the logarithms of numbers directly. The only step in this direction is the invention of the "log-log" slide-rule, which is very limited in its range, and the accuracy of its results may be open to question. The prospect of abolishing even multiplication and division from all ordinary calculations, and so making addition and subtraction the only necessary arithmetical processes, was sufficient inducement to the author to construct such a Table.

Before describing the difficulties that had to be overcome, a few words on the proposed nomenclature are necessary. In the first place, the word "logarithm" is unpronounceable and too long, which most people realise and avoid by calling it "log"; secondly, the inverse function has been very clumsily termed "anti-logarithm"; and, thirdly, the logarithm of the logarithm has been given the cacophonous name "log-log." It, therefore, seems permissible to devise more convenient names, constructed on some system. It is proposed to replace logarithm by "log," and to call the inverse function the "illog." The name "log-log" would sound better if reduced to "lolog," and its inverse function could then be systematically named the "illolog."

There are three difficulties encountered in constructing a lolog table:—

(1) The lolog of unity is infinite, and the differences in that region are very large.

(2) A base cannot be chosen so that all numbers having the same sequence of digits have the same mantissæ.

(3) The logs of numbers less than unity are negative, so that the lologs of such numbers are the logs of negative numbers.

The first of these difficulties is surmounted by diminishing the intervals at which the values are given in the neighbourhood of unity, and the second by limiting the range of the Tables. The third difficulty is avoided by neglecting the negative sign, which is, after all, an external feature that does not affect the numerical result of multiplication or division. For example, $\pm 2 \times \pm 3 = \pm 6$. Whatever the signs of these two factors, the numerical part of the result could be obtained by adding log 2 to log 3. The sign, however, would have to be determined independently.

If the lolog of 0·25 is required, one proceeds as follows :—

$$\log 0{\cdot}25 = \bar{1}{\cdot}39794 = -0{\cdot}60206.$$

Therefore $$\operatorname{lolog} 0{\cdot}25 = \log 0{\cdot}60206 = \bar{1}{\cdot}77964;$$

but it also happens that

$$\operatorname{lolog} 4{\cdot}0 = \log 0{\cdot}60206 = \bar{1}{\cdot}77964.$$

If the number $\bar{1}{\cdot}77964$ were given, the difficulty is to know whether its illolog is 4·0 or 0·25. The neglect of the sign of the log does not mean that merely the sign of the result has to be settled, but the numerical part is affected also. Although a little thought would always decide which is the correct value, the need for such thought would lead to errors and uncertainty, which would make the tables unsuitable for general use. All uncertainty of this nature can be overcome by printing the lologs of numbers less than unity on red paper, and of numbers greater than unity on white paper. The illologs would similarly be printed on red and white paper.

It is clear from the example above that the two numbers which have numerically equal lologs are reciprocals. This property enables reciprocals to be found very readily from lolog and illolog tables.

Let us now consider a simple case of involution and evolution. If it is given that $A^B = C$, then $A = \sqrt[B]{C}$.

Taking logs twice, $$\operatorname{lolog} C = \log B + \operatorname{lolog} A,$$

or, $$\operatorname{lolog} A = \operatorname{lolog} C - \log B.$$

From these two equations two simple rules can be derived :—

(1) To raise a number to the nth power, add the log of n to the lolog of the number, and the illolog of this sum is the desired result. The illolog must be found on a page of the same colour as that on which the lolog was found.

(2) To extract the nth root of a number, subtract the log of n from the lolog of the number, and the illolog of this difference is the desired result.

The illolog must be found on a page of the same colour as that on which the lolog was found.

To obviate mistakes almost entirely, it is recommended that lologs taken from red pages should be written down in red ink, also the result after adding or subtracting a log to or from a red lolog.

It has already been shown that the lologs of reciprocals are equal in magnitude, though different in colour. For this reason, expressions of the form

$$x^{-a} = \frac{1}{x^a} = \left(\frac{1}{x}\right)^a$$

can be evaluated with no more labour than that necessary to evaluate x^a. The rule is:—

(3) When performing a process of involution or evolution on the reciprocal of a number, merely change the colour of the lolog of the number itself, and proceed exactly as stated in Rules 1 and 2.

In the space available, it has not been possible to consider other properties of lologs which, though interesting, have little practical importance.

Involution and evolution of numbers and their reciprocals are of such frequent occurrence that the author ventures to hope the tables in question will prove useful in many different classes of calculation.